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Experiments on collisions between solitary waves

By T. MAXWORTHY

Departments of Aerospace and Mechanical Engineering,
University of Southern California, Los Angeles

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Experiments on 'head-on' collisions between two solitary waves show that the waves reach a maximum amplitude greater than twice the initial wave amplitude and that they suffer a time delay during their interaction. These results are compared with available theories and found to be in qualitative but not quantitative agreement.

1. Introduction

Maxworthy & Redekopp (1976) have recently proposed a new explanation for many of the features seen in Jupiter's atmosphere based on the properties of solitary waves in a horizontally sheared, stratified atmosphere on a β-plane. Such waves have a horizontal structure that obeys the modified Korteweg–de Vries (KdV) equation, an equation which is known to have solutions with many interesting features; in particular, solitary waves of very simple form are possible (see, for example, Hirota 1972). Much of the work on this subject has been reviewed by Scott, Chu & McLaughlin (1973), and the interested reader is advised to go there and to Whitham (1974) for some of the background material.

One of the major interests in the Jovian case is in the types of interaction that can occur between solitary waves, especially in reality, where dissipation and substantial 'vertical' accelerations can exist. Since shallow-water waves are known to obey the closely related KdV equation, it was decided to study such interactions in shallow water as a logical first step towards an understanding of the atmospheric case. The simplest of these interactions, from an experimental point of view at least, is the direct, head-on collision between two waves of equal amplitude travelling in opposite directions. Since the system is symmetric about the mid-plane it can be modelled by a single wave hitting a vertical end wall (but see below for comments on this suggestion). In §2 we describe the apparatus needed to perform this experiment and in §3 the results.

There have been a number of attempts to study the problem theoretically to various degrees of approximation and using a variety of methods. In §4 we review these theories and compare the predictions made by or inferred from them with the results of the present experiment.

2. Apparatus and procedure

The interaction was studied in two modifications of a single wave tank (5 m long, 20 cm wide and 30 cm deep). Water depths h_0 between 4·5 and 6·7 cm

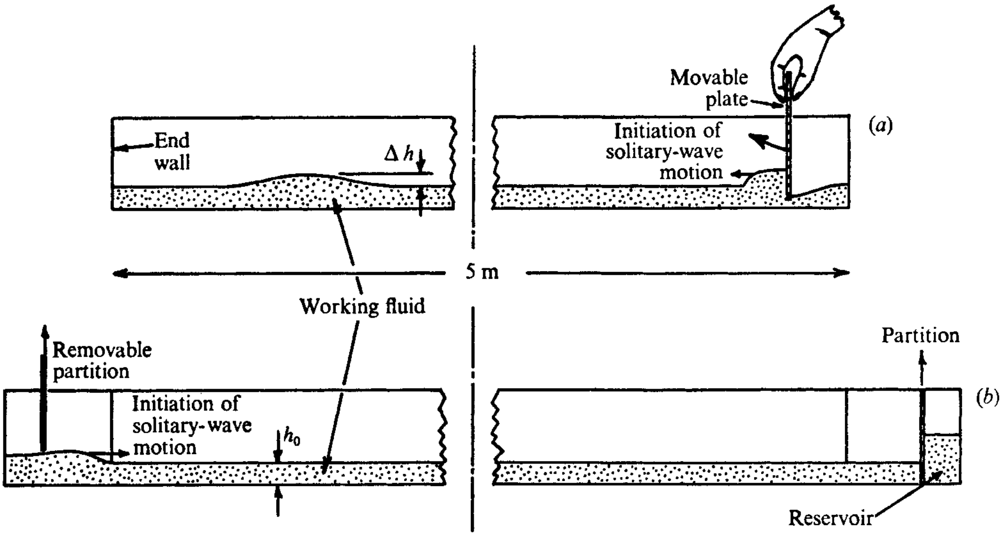

FIGURE 1. (a) Wave tank used to observe solitary-wave reflexions, showing method of wave production. (b) Same tank as in (a) but with extended ends and partitions to contain dammed-up fluid which, when released, produced solitary waves.

were used. A composite diagram of the apparatus is shown in figure 1. In order to study wave reflexion from a vertical wall, waves were produced by pulling a flat plate through the tank. Since it is well known that any initial disturbance will result in a sequence of ordered solitary waves plus a dispersive wave train (Segur 1973), one can, by judiciously varying the plate amplitude and velocity, produce single or multiple solitary waves with a minimum of other wave disturbances (wave amplitudes Δh of up to $0{\cdot}5\ h_0$ were used). Because the results obtained in this case were thought to be unusual at the time, it was suggested by several colleagues that perhaps the wave-reflexion case was not the same, in some subtle way, as the case of two waves interacting directly. As a result, two end tanks with movable partitions were added to each end of the long tank and waves were produced by their emergence from an initial square pulse (figure 1b). Again, by adjusting the length and height of the dammed-up fluid, sequences of one, two or more solitary waves could be produced at each end. They then propagated towards each other to interact at the centre of the tank.

Date were taken photographically at 64 frames/s and reduced by projecting onto a screen and measuring displacements, vertical heights and wave shapes by hand. Scales of length and time could be placed in the field of view of the camera in order to calibrate the system.

3. Results

In figure 2 we show the most significant result of these experiments, for a typical initial wave amplitude. The spatial phase shift in the wave trajectories is negative, that is the reflected wave appears to have come from a virtual origin behind the real wall or alternatively it appears to hesitate at the wall and reflects

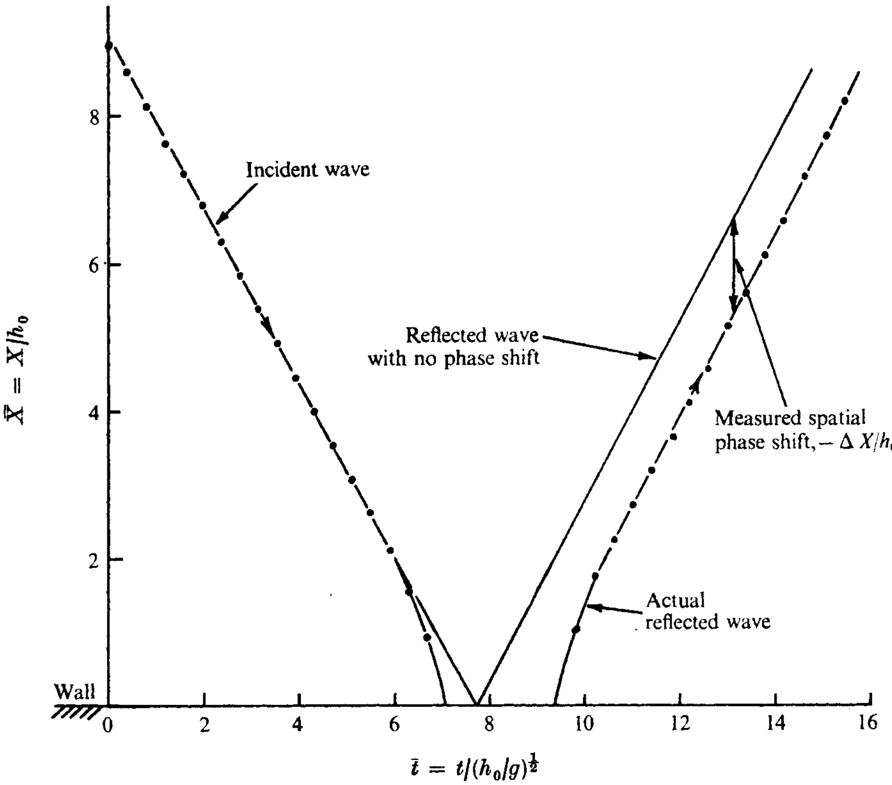

FIGURE 2. Typical wave trajectory, i.e. motion of point of maximum amplitude, in dimensionless co-ordinates, showing spatial phase change of the wave crest due to finite time of interaction at the wall. $\Delta h/h_0 = 0{\cdot}31$. Speed of incident wave: theoretical value = 92·55 cm/s; measured value = 93 cm/s.

such that it reaches a point downstream delayed in time. Figure 3 shows the magnitude of the phase shift for a variety of initial depths and wave amplitudes and for the two cases of direct wave–wave interaction and wave reflexion. From this figure it is clear that the magnitude of the phase shift is independent of the wave amplitude and does not depend on the type of interaction, within the large experimental error. Unfortunately, waves of very small amplitude could not be measured, so that it is not possible to say whether or not the curve remains at a finite level as one might suspect on elementary grounds (see §4) or whether it tends to zero phase shift for very small amplitudes, as available theories suggest.

In figure 4 we see that the maximum amplitude attained by the wave during interaction is always greater than twice the initial wave amplitude. Here we note that this amplitude does tend to zero as the initial amplitude tends to zero and that, because of viscous and wetting effects at the wall, this case has a maximum amplitude less than that for wave–wave interaction. This latter case also contains effects due to surface tension at the wave peak, which creates a rather distorted surface that adds to the discrepancy between the two results (see §4).

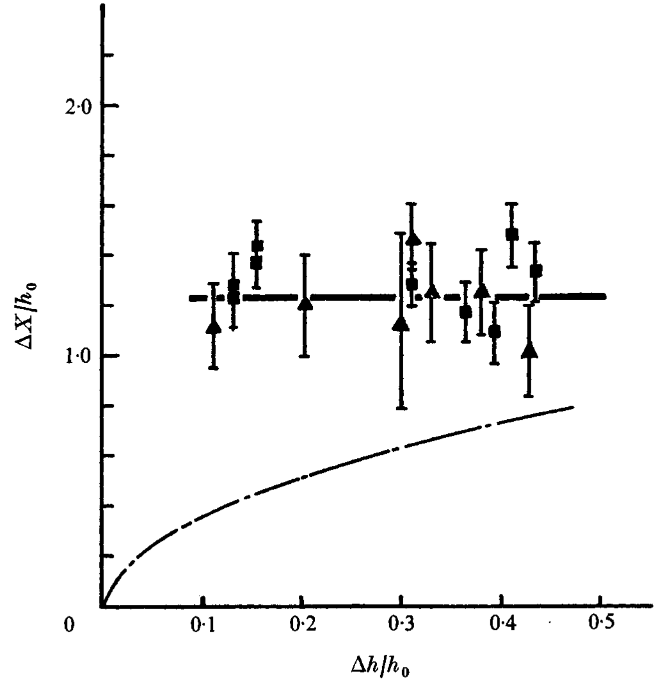

FIGURE 3. Magnitude of phase shift $\Delta X/h_0$ *vs.* wave amplitude $\Delta h/h_0$ for several values of h_0 and for both types of interaction, i.e. wave–wave interaction (triangles) and end-wall reflexion (squares). — - —, Oikawa & Yajima (1973) and implicit result from Byatt-Smith (1971).

Details of typical interactions at a moderate amplitude of $\Delta h/h_0 = 0{\cdot}31$ are shown in figure 5. Several points are worth noting. First, the incoming wave shape is very closely approximated by the classical expression (figure 5*a*); it is imperceptibly steeper, an effect which is, presumably, accounted for by higher-order corrections to the calculated profile. During the interaction with the wall (figures 5*b*–*g*) vertical fluid accelerations are large and any theory that purports to describe the motion must take them into account (see §4 for a detailed discussion). The reflected wave (figures 5*h*, *i*) assumes a shape that is clearly steeper than that of the incoming wave and is, in fact, moving slightly faster (see figure 2). There is also a clear indication of a second, weaker wave following the first, an effect which is enhanced at higher amplitudes (figure 6).

4. Discussion and conclusions

Comparison of the present results with available theoretical efforts is hampered by the somewhat confused state of the latter. In what follows we present what seems to be the current state, realizing that more analysis needs to be done. A first fundamental concern is with the equation or set of equations to be used to describe the interaction under consideration. One commonly used and basic pair of equations (see, for example, Byatt-Smith 1971, equations (1.2) and (1.3))

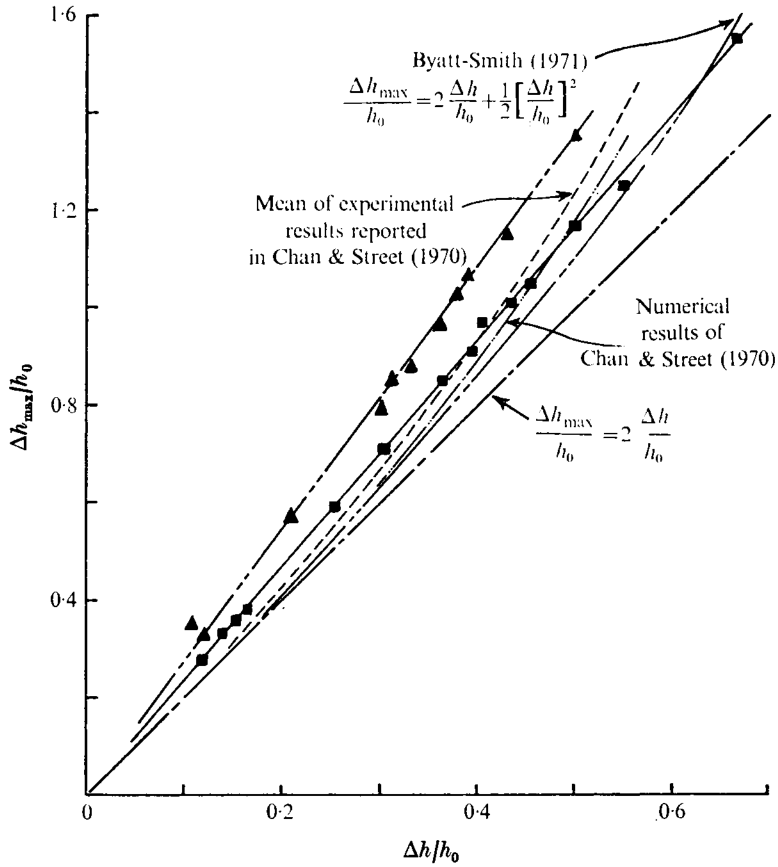

FIGURE 4. Maximum amplitude $\Delta h_{max}/h_0$ attained by wave during interaction *vs*. initial wave amplitude $\Delta h/h_0$. ▲, wave–wave interaction; ■, end-wall reflexion. Also included are the results of Byatt-Smith (1971) and experimental and numerical values reported in Chan & Street (1970)

is due to Boussinesq (1872), who also combined them into one equation (Byatt-Smith 1971, equation (1.4)), which has often been interpreted as being suitable for describing waves travelling in two directions. However, as pointed out by Long (1964), one fundamental assumption made during the simplification is that the waves travel in only one direction. Also, Byatt-Smith (1971) has shown that the single equation ignores unsteady terms of order $(\Delta h/h_0)^2$ which are present in the pair of equations. Thus any solution based on the reduced equation is unlikely to be correct, and in fact, Hirota (1973) and the first solution presented in Oikawa & Yajima (1973) predict a phase shift of opposite sign to that found experimentally and a maximum amplitude less than twice the initial wave height!

The first solution to a corrected, single equation appears to have been given by Byatt-Smith (1971), who obtained an implicit result for the phase shift which is in agreement with the explicit result obtained later by Oikawa & Yajima (1973), from the pair of Boussinesq equations (Dr P. D. Weidman, private

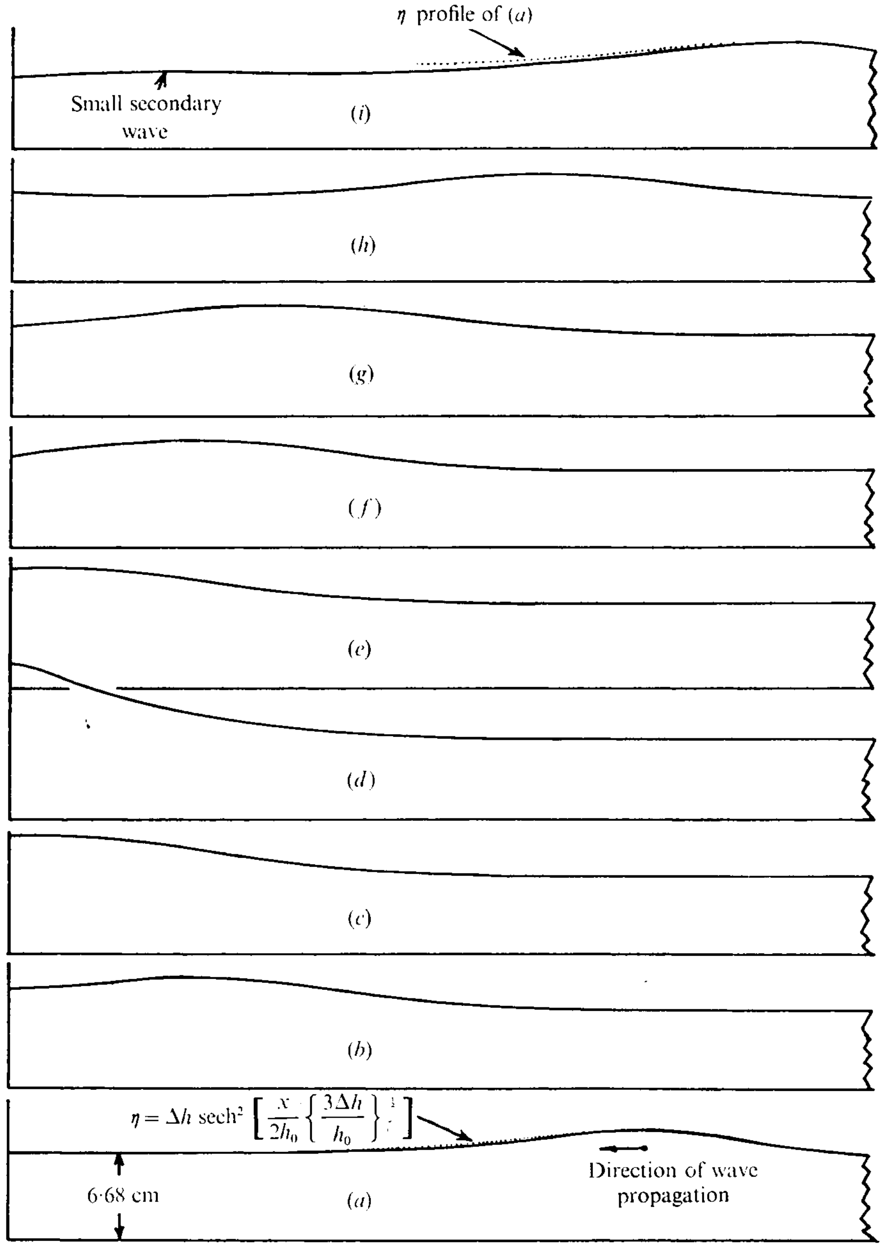

FIGURE 5. Wave profiles traced from a moving-picture sequence of the interaction of a single wave with the tank end wall for a moderate initial amplitude, $\Delta h/h_0 = 0{\cdot}31$. (*a*) 0 s. (*b*) 0·35 s. (*c*) 0·43 s. (*d*) 0·54. (*e*) 0·67 s. (*f*) 0·73 s. (*g*) 0·80 s. (*h*) 1·02 s. (*i*) 1·19 s. Note the change in amplitude from (*c*) to (*d*) to (*e*) in a very short time.

communication). This result is plotted in figure 3. Byatt-Smith did obtain an explicit expression for the maximum height reached by the wave and this is plotted on figure 4. In both cases the more complete single equation (derived by Byatt-Smith 1971) or the set of equations gives results that are in qualitative agreement with the experiments; i.e. the phase shift is in the observed direction

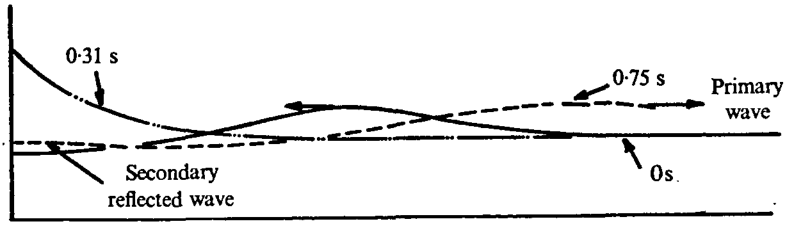

FIGURE 6. As figure 5 for a large initial amplitude, $\Delta h/h_0 = 0{\cdot}39$, showing the production of secondary waves after the main interaction.

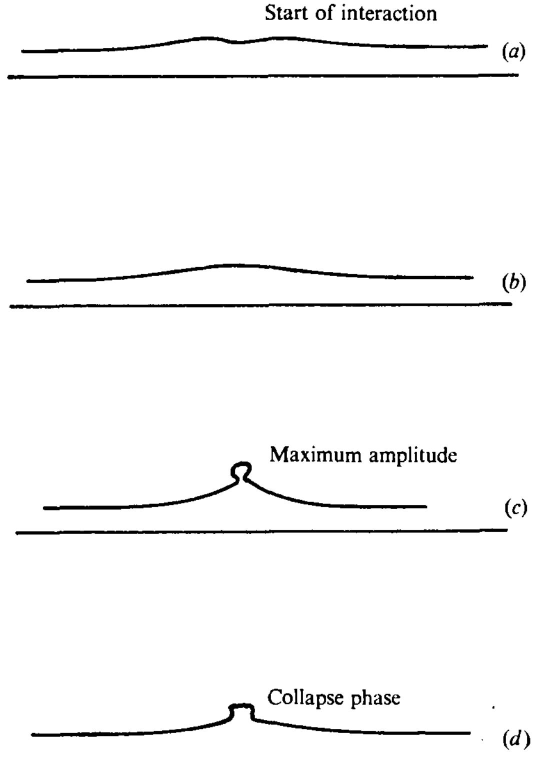

FIGURE 7. Sketch from an actual experiment showing the distorted wave peak formed during a wave–wave interaction at large initial amplitude.

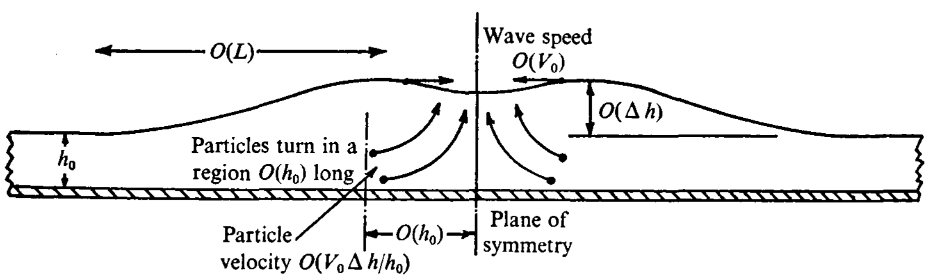

FIGURE 8. Sketch of wave–wave interaction showing stagnation of the horizontal particle velocity and production of a vertical velocity that elevates the fluid surface in a time of order h_0/V_0.

and the maximum amplitude is greater than twice the initial amplitude. Also shown in figure 4 are some experimental results and a numerical solution presented by Chan & Street (1970), who, unfortunately, did not measure or calculate the phase shift. Had they done so the present work would have been superfluous. We suspect that the small difference between their results and the present experiments, for wave interaction with a vertical wall, are due to a combination of effects: first, at small initial amplitudes, experimental errors due to our inability to define precisely the wave surface because of the effects of a variable side-wall wetting angle and second, at large initial amplitudes, the relatively greater importance of viscous and end-wall wetting effects in our smaller tank. It is clear that the maximum amplitude reached by the wave right at the end wall is very sensitive to the latter effects, which are absent in our wave–wave interaction experiments. As a result it is tempting to say that the latter results are the correct ones. Unfortunately this statement is weakened by the observation that, especially at large initial amplitudes, the vertical accelerations tend to create a jet-like flow at the wave peak which breaks down into individual drops, as sketched in figure 7. Because none of these extraneous effects are included in the theories, the comparisons shown on figure 4 are probably not too meaningful and the linear dependence on $\Delta h/h_0$ not to be taken too seriously.

Finally, there is a theory due to Benney & Luke (1964), which has been criticized in Byatt-Smith (1971) and which predicts zero phase shift (Dr L. G. Redekopp, private communication) and a maximum height less than $2\Delta h$.† Regrettably this theory, although correctly formulated, appears to contain algebraic errors† and will, presumably, agree with the other available theories when corrected. In all these theories the initial and final wave shapes are the same with no indication of the production of the secondary waves found experimentally.

Thus, although some of the theories are in qualitative agreement with our experimental results there are quantitative differences. In particular, our discovery of a phase shift that is independent of initial amplitude seems to be significant and deserves some explanation. As a result we have constructed a simple order-of-magnitude argument that gives the correct result and contains the essential physics of the problem.

Consider a wave of small amplitude Δh propagating on a shallow fluid layer of depth h_0, so that its wave speed is $O(gh_0)^{\frac{1}{2}} = O(V_0)$. Associated with the wave motion is a horizontal particle velocity of order $V_0 \Delta h/h_0$. When two such waves interact 'head-on' this horizontal motion is brought to rest and creates a vertical velocity also of order $V_0 \Delta h/h_0$ (see figure 8 for a diagrammatic view of this process). The last estimate is based solely on the likelihood that the stagnation and turning process takes place in a region that is $O(h_0)$ in both vertical and horizontal extent as in classical stagnation-point flow (see for example Rosenhead 1963, p. 155). With this velocity estimate, the time taken for the wave to peak and return to its original position is $O(h_0/V_0)$ and thus finite for all amplitudes. The associated spatial phase delay is $O(h_0)$, as we have experimentally! There now seem to be

† Comments of a referee.

at least two possible reasons for the discrepancy between our experiment (and the order-of-magnitude argument given above) and the available theories. First, because of the large vertical accelerations that exist during the real interaction, the Boussinesq equations are inadequate and a new scaling procedure should be adopted to include higher-order effects that are at present ignored. Second, if the present equations are adequate one needs many more terms in the expansion before the experimental results are approached. Of the two possibilities the first appears to be the most likely.

Unfortunately, our order-of-magnitude arguments cannot be used to decide why the maximum amplitude during interaction is greater than twice the pre-interaction amplitude. This is because the amplitude is already of the correct order of magnitude, and the result depends on the subtle adjustment of the waves to their new condition and not on any gross, new dynamical feature.

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